SOME POSSIBLE ARRANGEMENTS OF PARAMETRIC AMPLIFIERS EMPLOYING LOWER FREQUENCY PUMPING

N. B. CHAKRABARTI† AND K. D. DIKSHIT

J. K. INSTITUTE OF APPLIED PHYSICS, UNIVERSITY OF ALLAHABAD, ALLAHABAD

(*Received, August* 16, 1959)

ABSTRACT. An analysis of certain parametric amplifiers using lower frequency pumping in lumped constant circuits is presented. Two cases (i) combination of a mixer and an amplifier using one pump and two idlers and (ii) combination of a mixer and an amplifier using two pumps and two idlers, have been treated in detail. The phase and power relations at signal frequency, pump and idling frequencies have been discussed. The expressions for negative resistance, gain, band-width and noise figure for each case have been derived

Two other possible cases have been mentioned. It is shown that the multi-idler circuits offer no added advantage.

I. INTRODUCTION

In recent years good deal of work has been done on parametric amplifiers which are of great importance in low noise work. In most of the work done. (Bloom and Chang, 1957, Heffner and Wade, 1958) higher frequency pumping requiring pumping power at a frequency higher than that of the signal, has been utilised for signal amplification. The limitation of such amplifiers is that of power at higher frequencies. In the centimetric region, in particular, it would be good to be able to use a lower frequency pumping source. One case of lower frequency pumping has been treated by Chang and Bloom (1958) wherein they have employed two pumps and an idler. They have used reversed-biased junction diodes or nickel-manganese ferrite exhibiting non-linearity of the cubic order as a non-linear coupling reactance.

In this paper some possible arrangements of parametric amplifiers using lower frequency pumping in lumped constant circuit arrangements and employing quadratic non-linearity of the coupling reactance are suggested. Those include (i) combination of a mixer and an amplifier using one pump and two idlers and (ii) combination of a mixer and an amplifier using two pumps and two idlers. The phase and power relations at signal frequency, idler and pump frequencies

* For an excellent bibliography see Bloom and Chang (1957).

† Now at Institute of Radio Physics and Electrons, Calcutta.

have been discussed. Expressions for negative resistance, gain band-width and noise figure have been derived.

II. PARAMETRIC AMPLIFICATION*

It is known that energy can be extracted from a source driving an energy storage element such as an inductor or a capacitor and fed to the fields of a resonant circuit which is suitably coupled to the energy storage element. This fact can be used for amplifying signals. The amplifiers based on this principle are called variable parameter or parametric amplifiers, because here the amplification is achieved by the variation of a parameter of the system.

The principle of a parametric amplifier can be best understood by considering the case of higher frequency pumping In the system of figure 1, a variable

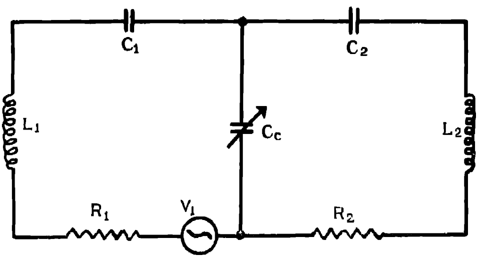

Fig. 1 Schematic representation of a single pump single idler parametric amplifier having a variable non-linear capacitor as the coupling element

non-linear reactance couples two series resonant circuits—one called the signal circuit having its angular resonant frequency ω_1 and other called the idler having its angular resonant frequency $\omega_2 = \omega_1 + \omega_p$, where ω_p is the angular frequency of the pumping source driving the non-linear element. The power at the frequency ω_p mixes with that at ω_1 and causes a current at the idling frequency ω_2 to flow in the coupling reactance. The flow of power at the idling frequency throws a negative resistance to the signal circuit Amplification of the signal is thus achieved.

The parametric amplifiers using lower frequency pumping, to be treated here, can be considered as a combination of mixers and amplifiers. We shall consider only the following cases :

(i) combination of a mixer and an amplifier using one pump and two idlers (Hogan *et al*, 1958).

(ii) combination of a mixer and an amplifier using two pumps and two idlers,

(iii) combination of two mixers and one amplifier using two pumps and two idlers and

(iv) combination of one mixer and two amplifiers using two pumps and three idlers.

Other arrangements employing more idlers do not offer additional advantages. We shall treat the first two cases in detail and shall briefly discuss the other two cases

III. ANALYSIS OF CASE I

(1) *Phase Relations* . Let us consider the case of a mixer and an amplifier with frequency relations

$$\omega_1 = \omega_2 + \omega_p \qquad \text{(mixer)} \qquad \ldots \quad (1.a)$$

$$\omega_p = \omega_2 + \omega_3 \qquad \text{(amplifier)} \qquad \ldots \quad (1.b)$$

The subscript 1 denotes signal. subscripts 2, 3 and 4 denote idlers and the subscripts p and q denote the pumps. The coupling reactance taken is a non-linear inductor. The analysis would, however, apply equally well to a system employing a non-linear capacitor as the coupling element.

The idlers ω_2 and ω_3 take power from pumping source ω_p, ω_2 in turn combines with ω_p to supply power at the signal frequency ω_1.

Consider the resonant circuit shown in figure 2 and suppose that :

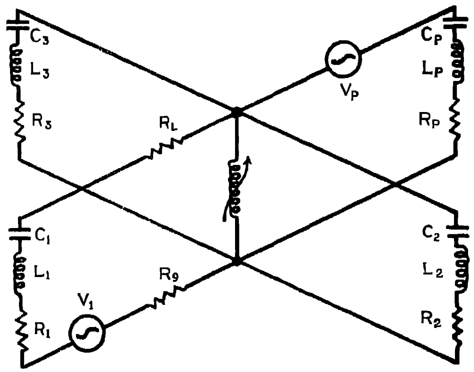

Fig. 2. Schematic representation of a single pump two idler parametric amplifier having a variable non-linear inductor as the coupling element.

L_K = inductance of the k-th circuit,

C_K = capacitance of the k-th circuit,

R_2, R_3, R_p = resistances of the 2nd, 3rd and p-th circuit respectively,

R_1 = coil resistance of the signal circuit,

R_L = load resistance,

R_g = internal resistance of the signal source,

$V_1(e^{j\omega_1 t}+e^{-j\omega_1 t})$ = voltage generated by the signal source.

and $V_p(e^{j\omega_p t}+e^{-j\omega_p t})$ = voltage generated by the pump source.

The coupling reactor is an inductor with quardratic non-linearity, that is, the flux varies with the instantaneous current $i_{(t)}$ as given by equation (2)

$$\phi = L_0 i_{(t)} - L i_{(t)}^2 \quad \ldots (2)$$

where

L = coefficient of non-linearity,

L_0 = linear portion of the inductance,

$i_{(t)}$ = total instantaneous current at time t

$$=\sum_{k=1,2,3,p} (I_K e^{j\omega_k t} + I_K^* e^{-j\omega_k t}),$$

*denotes the conjugate.

The voltage across the reactor at any instant will be given by

$$v_{(t)} = \frac{d\phi}{dt} = L_0 \frac{di_{(t)}}{dt} - 2L i_{(t)} \frac{di_{(t)}}{dt}$$

$$= L_0 \sum_{k=1,2,3,p} j\omega_k (I_k e^{j\omega_k t} - I_k^* e^{-j\omega_k t})$$

$$-2L. \sum_{k=1,2,3,p} j\omega_k (I_k e^{j\omega_k t} - I_k^* e^{-j\omega_k t}) \times \sum_{k=1,2,3,p} (I_k e^{j\omega_k t} + I_k^* e^{-j\omega_k t}). \quad \ldots (3)$$

If we designate the voltage at a frequency ω_k as v_k, the various components of the voltages at different frequencies will be

$$v_1 = j\omega_1 L_0 I_1 e^{j\omega_1 t} - j2\omega_1 L I_3 I_p e^{j\omega_1 t} \quad \ldots (4.a)$$

$$v_2 = j\omega_2 L_0 I_2 e^{j\omega_2 t} - j2\omega_2 L[I_1 I_p^* + I_p I_3^*] e^{j\omega_2 t} \quad \ldots (4.b)$$

$$v_3 = j\omega_3 L_0 I_3 e^{j\omega_3 t} - j2\omega_3 L I_p I_2^* e^{j\omega_3 t} \quad \ldots (4.c)$$

$$v_p = j\omega_p L_0 I_p e^{j\omega_p t} - j2\omega_p L[I_1 I_2^* + I_2 I_3] e^{\omega_p t} \quad \ldots (4.d).$$

Supposing the idler circuit (3) to be resonant at ω_3, we have

$$R_3 I_3 = j2\omega_3 L I_p I_2^* \quad \ldots (5)$$

Let us write $I_k = I_k' e^{j\theta k}$

where

θ_k = the phase difference between the applied voltage and the resulting current.

We now have

$$R_3 I_3' e^{j\theta_3} = j2\omega_3 L I_p' I_2' e^{j(\theta_p - \theta_2)}$$

Hence the phases will automatically be adjusted such that

$$\theta_3 = \theta_p - \theta_2 + \pi/2 \quad \ldots \quad (6)$$

Similarly for the idler circuit (2) resonant at ω_2.

$$R_2 I_2' e^{j\theta_2} = j2\omega_2 L\, I_p' [I_1' e^{j(\theta_1 - \theta_p)} + I_3' e^{j(\theta_p - \theta_3)}]$$

Using equation. (6)

$$R_2 I_2' e^{j\theta_2} = 2\omega_2 L I_p' [I_1' e^{j(\theta_1 - \theta_p + \pi/2)} + I_3' e^{j\theta_2}] \quad \ldots \quad (7)$$

This requires the phases to be adjusted such that

either $\quad \theta_1 - \theta_p = \theta_2 + \pi/2$ and $I_3' > I_1'$ if $I_2' R_2 < 2\omega_2 L I_p' I'_3$ ...(8.a)

Or $\quad \theta_1 - \theta_p = \theta_2 - \pi/2$ if $I_2' R_2 > 2\omega_2 L I_p' I_3'$ ... (8.b)

Taking relation (8.a)

$$v_1 = j\omega_1 L_0 I_1' e^{j(\omega_1 t + \theta_1)} - j2\omega_1 L_2' I_p' e^{j(\omega_1 t + \theta_1 - \pi/2)}$$

$$= j\omega_1 L_0 I_1' e^{j(\omega_1 t + \theta_1)} - 2\omega_1 L I_2' I_p' e^{j(\omega_1 t + \theta_1)}$$

The negative sign before the second term in the above equation indicates that the power should be given to the signal circuit. In other words the signal power should be amplified at the cost of the pump power. On the other hand if the phase relations are given as in eqn. (8.b), the power should be extracted from the source and the signal would be attenuated instead of being amplified. Thus the amplification can be achieved only if

$$R_2 I_2' < 2\omega_2 L\ I_p' I_3' \qquad \{\text{vide equation. (8.a)}\}$$

Substituting the value of I_3', the condition of amplification becomes

$$R_2 < \frac{4\omega_2\omega_3 L^2 I'^2_p}{R_3} \quad \ldots \quad (9)$$

This is verified in section III(3).

(2) *Power Relations* : The last terms on the right hand side of the equations

(4) give the powers at different frequencies entering the coupling reactor. Denoting the power entering the coupling reactor at frequency ω_k as P_k, we find

$$P_1 = 2L\omega_1 |I_1 I_2 I_p| \qquad \dots \quad (10.a)$$

$$P_2 = -2L\omega_2[|I_1 I_2 I_p| + |I_2 I_3 I_p|] \qquad \dots \quad (10.b)$$

$$P_3 = -2L\omega_3 |I_2 I_3 I_p| \qquad \dots \quad (10.c)$$

$$P_p = -2L\omega_p[|I_1 I_2 I_p| - |I_2 I_3 I_p|] \qquad \dots \quad (10.d)$$

From equations (10), we obtain

$$\frac{P_p}{\omega_p} - \frac{P_2}{\omega_2} + \frac{2P_3}{\omega_3} = 0 \qquad \dots \quad (11.a)$$

$$\frac{P_1}{\omega_1} + \frac{P_2}{\omega_2} - \frac{P_3}{\omega_3} = 0 \qquad \dots \quad (11.b)$$

These are the Manley-Rowe (1956) relations for this case

(3) *Negative Resistance*. The signal frequency will not awlays be ω_1, instead it may be, in general. ω'_1 where $\omega'_1 = \omega_1 + \Delta\omega$. The idling frequencies will then be

$$\omega'_2 = \omega_2 + \Delta\omega \quad \text{and} \quad \omega'_3 = \omega_3 - \Delta\omega$$

Under the assumption that the Q's of the circuits are quite high and the frequencies are well separated, we can write the general voltage current relations as given below

$$V_1 = Z_1 I_1 - j2\omega_1 L I_2 I_p \qquad \dots \quad (12.a)$$

$$0 = Z_2 I_2 - j2\omega_2 L[I_1 I_p^* + I_3^* I_p] \qquad \dots \quad (12.b)$$

$$0 = Z_3 I_3 - j2\omega_3 L I_2^* I_p \qquad \dots \quad (12.c)$$

$$V_p = Z_p I_p - j2\omega_p L[I_2 I_3 + I_1 I_2^*] \qquad \dots \quad (12.d)$$

where

$$Z_1 = R_T + jX_1 = (R_1 + R_L + R_g) + jX_1$$

$$= R_T + j\left[\omega'_1(L_0 + L_1) - \frac{1}{\omega_1' C}\right]$$

$$= R_T\left[1 + j2Q_1\frac{\Delta\omega}{\omega_1}\right] \qquad \dots \quad (13.a)$$

$$Z_2 = R_2\left[1 + j2Q_2\frac{\Delta\omega}{\omega_2}\right] \qquad \dots \quad (13.b)$$

$$Z_3 = R_3\left[1 - j2Q_3\frac{\Delta\omega}{\omega_3}\right] \qquad \dots \quad (13.c)$$

$$Z_p = R_p\left[\ 1 + j2Q_p\frac{\Delta\omega}{\omega_p}\right] = R_p \qquad \dots \quad (13.d)$$

The effective self impedance Z_{11} for the signal circuit can be obtained by eliminating idling currents from equation. (12.a) and will be given by equation (14)

$$Z_{11} = \frac{V_1}{I_1}$$

$$= Z_1 - \frac{4\omega'_1\omega'_2L^2|I_p|^2}{\frac{4\omega'_2\omega'_3L^2|I_p|^2}{Z_3^*} - Z_2} \qquad \dots \quad (14)$$

From this we infer that in order to obtain negative resistance and hence amplification of the signal, it is necessary that

$$Z_2 < \frac{4\omega'_2\omega'_3L^2|I_p|^2}{Z_3^*}$$

and at resonance

$$R_2 < \frac{4\omega_2\omega_3L^2|I_p|^2}{R_3} \qquad \dots \quad (15)$$

which is in confirmity with the condition {equation. (9)} of amplification obtained from phase consideration in section III(1). Equation. (14) can be written as

$$Z_{11} = (R_T - R) + j(X_1 - X) \qquad \dots \quad (16)$$

where the negative resistance R and the reactance X are given by equations, (17)

$$R = \frac{R_TQ_1Q_2\beta_{1p}\beta_{2p}\left(Q_2Q_3\beta_{2p}\beta_{3p} - 4Q^2{}_3\frac{\Delta\omega^2}{\omega_3{}^2} - 1\right)}{\left(Q_2Q_3\beta_{2p}\beta_{3p} + 4Q_2Q_3\frac{\Delta\omega^2}{\omega_2\omega_3} - 1\right)^2 + 4\Delta\omega^2\left(\frac{Q_2}{\omega_2} + \frac{Q_3}{\omega_3}\right)^2} \qquad \dots \quad (17a)$$

$$X = \frac{2\Delta\omega R_TQ_1Q_2\beta_{1p}\beta_{2p}\left(\frac{Q_2}{\omega_2} + \frac{Q_2Q^2{}_3\beta_{2p}\beta_{3p}}{\omega_3} + \frac{4Q_2Q^2{}_3\Delta\omega^2}{\omega_2\omega^2{}_3}\right)}{\left(Q_2Q_3\beta_{2p}\beta_{3p} + 4Q_2Q_3\frac{\Delta\omega^2}{\omega_2\omega_3} - 1\right)^2 + 4\Delta\omega^2\left(\frac{Q_2}{\omega_2} + \frac{Q_3}{\omega_3}\right)^2} \qquad \dots \quad (17.b)$$

where

$$\beta_{kp} = 2\ \frac{L|I_p|}{L_k}\cdot\frac{\omega'_k}{\omega_k} \qquad \dots \quad (18)$$

At resonance i.e. for $\Delta\omega = 0$, the negative resistance becomes

$$R_{res} = \frac{4\omega_1\omega_2L^2|I_p|^2}{\frac{4\omega_2\omega_3L^2|I_p|^2}{R_3} - R_2} \qquad \dots \quad (19.a)$$

or

$$R_{res} = R_T Q_1 Q_2 \beta_{1p} \beta_{2p} / (Q_2 Q_3 \beta_{2p} \beta_{3p} - 1) \quad \ldots \quad (19.b)$$

(4) *Gain and Band-width* : The power gain of the amplifier is defined as

$$G = \frac{P_{ou}}{P_{in}} = \frac{R_L |I_1|^2}{|V_1|^2/4R_g} \quad \ldots \quad (20.a)$$

Substituting the value of $|I_1|^2/|V_1|^2$ from equation. (16)

$$G = \frac{4R_g R_L}{|Z_{11}|^2} = \frac{4R_g R_L}{(R_T - R)^2 + (X_1 - X)^2} \quad \ldots \quad (20.b)$$

The gain will be maximum at resonance and will be given by

$$G_{max} = 4R_g R_L / (R_T - R_{res})^2 \quad \ldots \quad (21)$$

The normalized gain ($= G/G_{max}$) can be written as

$$\frac{G}{G_{max}} = \left[\frac{R_T - R_{res}}{|Z_{11}|} \right]^2 \quad \ldots \quad (22)$$

The band-width of the amplifier will be equal to the difference of the roots of the equation.

$$\frac{G}{G_{max}} = \tfrac{1}{2} \quad \ldots \quad (23)$$

that is

$$2\left[1 - \frac{Q_1 Q_2 \beta_{1p} \beta_{2p}}{Q_2 Q_3 \beta_{2p} \beta_{3p} - 1} \right]$$

$$= \left[1 - \frac{Q_1 Q_2 \beta_{1p} \beta_{2p} \left(Q_2 Q_3 \beta_{2p} \beta_{3p} - 4 Q_3^2 \frac{\Delta\omega^2}{\omega_3^2} - 1 \right)}{\left(Q_2 Q_3 \beta_{2p} \beta_{3p} + 4 Q_2 Q_3 \frac{\Delta\omega^2}{\omega_2 \omega_3} - 1 \right)^2 + 4\Delta\omega^2 \left(\frac{Q_2}{\omega_2} + \frac{Q_3}{\omega_3} \right)^2} \right]^2$$

$$+ \left[2\Delta\omega \left\{ \frac{Q}{\omega_1} - \frac{Q_1 Q_2 \beta_{1p} \beta_{2p} \left(\frac{Q_2}{\omega_2} + \frac{Q_2 Q_3^2 \beta_{2p} \beta_{3p}}{\omega_3} + \frac{4 Q_2 Q_3^2 \Delta\omega^2}{\omega_2 \omega_3^2} \right)}{\left(Q_2 Q_3 \beta_{2p} \beta_{3p} + 4 Q_2 Q_3 \frac{\Delta\omega^2}{\omega_2 \omega_3} - 1 \right)^2 + 4\Delta\omega^2 \left(\frac{Q_2}{\omega_2} + \frac{Q_3}{\omega_3} \right)^2} \right\} \right]^2 \quad \ldots \quad (24)$$

If the value of Q's and β's be known, the value of $\frac{1}{|Z_{11}|^2}$ can be plotted against $\Delta\omega$ and the band-width can be determined.

It is obvious that the value of maximum gain and band-width for a particular value of inductance variation is entirely governed by the negative resistance. Larger is the value of R the larger is the gain. From expression (17.b) it is seen that R can be increased by increasing the value of Q_1 and Q_2 and decreasing the value of Q_3. The increased values of ω_2 and ω_3 will also have similar effect. The value of inductance variation required for annuling the total circuit resistance can be written from equation (19.b) as

$$Q_1Q_2\beta_{1p}\beta_{2p} = Q_2Q_3\beta_{2p}\beta_{3p}-1 \qquad \dots \quad (25.a)$$

or

$$2LI_p = \left(\frac{R_1 R_2 R_3}{\omega_2\omega_3R_1-\omega_1\omega_2R_3}\right)^{\frac{1}{2}} \qquad \dots \quad (25.b)$$

It is therefore seen that for the values of $2LI_p$ given by equationn. (26)

$$\left(\frac{R_2R_3}{\omega_2\omega_3}\right)^{\frac{1}{2}} \leqslant 2LI_p \leqslant \left(\frac{R_1R_2R_3}{\omega_2\omega_3R_1-\omega_1\omega_2R_3}\right)^{\frac{1}{2}} \qquad \dots \quad (26)$$

the system is unstable and the sustained oscillations will take place. In order to use the combination as an amplifier the adjustment is such that the value of $2LI_p$ approaches

$$\left(\frac{R_1 R_2 R_3}{\omega_2\omega_3R_1-\omega_1\omega_2R_3}\right)^{\frac{1}{2}}$$

but still

$$Q_1Q_2\beta_{1p}\beta_{2p} < (Q_2Q_3\beta_{2p}\beta_{3p}-1)$$

Such an adjustment will give maximum gain

Usually in practice Q_1, the loaded Q of the signal circuit, is quite small, that is, $Q_1Q_2\beta_{1p}\beta_{2p}$ is a small fraction and hence for large values of gain we can assume

$$(Q_2Q_3\beta_{2p}\beta_{3p}-1) < < 1 \qquad \dots \quad (27)$$

With this assumption, the band-width can approximately be written as

$$2\Delta\omega = \frac{Q_2Q_3\beta_{2p}\beta_{3p}-1}{\dfrac{Q_2}{\omega_2}+\dfrac{Q_3}{\omega_3}} \qquad \dots \quad (28)$$

(5) *Noise Figure* · We shall now find out the noise figure of the amplifier under the assumption that the signal to be amplified is precisely at resonant frequency ω_1 of the tank circuit (1). Noise figure is written as

$$F = \frac{S_i/N_i}{S_0/N_0}$$

$$= \frac{1}{\text{Power gain}} \times \frac{1}{KT_0\Delta f} N_0 \qquad \dots \quad (29)$$

where

$$\frac{S_i}{N_i} = \text{available signal to noise ratio at the input}$$

$$\frac{S_0}{N_0} = \text{available signal to noise ratio at the output}$$

K = Boltzmann's constant

T_0 = Standard noise temperature = 290°K, and

Δf = the noise band-width of the amplifier.

In order to calculate N_0, we return to original equations (12) and replace the signal voltages by noise voltages as shown in figure 3.

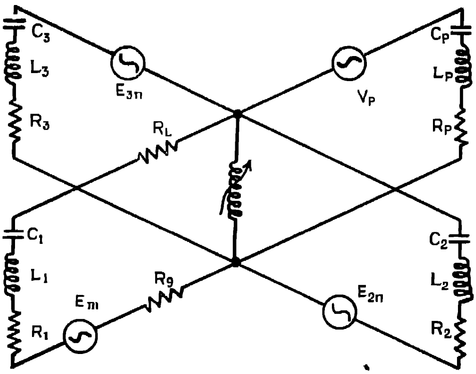

Fig. 3. Figure 2 redrawn to indicate noise sources.

We can find the individual contribution to noise power by any one circuit by putting all the noise voltages except the one under consideration, equal to zero and then eliminating the idling currents. The total* noise current square is

$$[I_1]^2_n = \sum_{k=1,2,3} [I_1]^2_{kn}$$

$$= \left[\left(\frac{E_{1n}}{|Z_1|-R}\right)^2 + \left(\frac{E_{2n}R}{2\omega_2 L|I_p|}\cdot\frac{1}{|Z_1|-R}\right)^2 + \left(\frac{E_{3n}}{|Z_3|}\cdot\frac{R}{|Z_1|-R}\right)^2\right] \quad \ldots \quad (30)$$

where

$[I_1]_{kn}^2$ = noise current squared in the k-th circuit

$$N_0 = \frac{R_L}{(|Z_1|-R)^2}\left[E_{1n}^2 + \left(\frac{E_{2n}R}{2\omega_2 L|I_p|}\right)^2 + \left(\frac{E_{3n}}{Z_3}\cdot R\right)^2\right]$$

* It may be noted that we have neglected the noise voltage E_{pn} due to R_p because it is very small as compared to the pump voltage V_p. We have also taken no account of the noise voltage due to the fluctuations of the pump circuit.

As E_{kn} are the thermal noise voltages, we can write

$$E_{1n}^2 = 4K\Delta f(R_g T_0 + R_1 T) \quad \dots \quad (31.a)$$

$$E_{2n}^2 = 4KT\Delta f R_2 \quad \dots \quad (31.b)$$

$$E_{3n}^2 = 4KT\Delta f R_3 \quad \dots \quad (31.c)$$

Substituting these values we get

$$N_0 = \frac{4K\Delta f R}{(|Z_1|-R)^2}\left[(R_g T_0 + R_1 T) + \frac{R_2 T R^2}{4\omega_2^2 L^2 |I_p|^2} + \frac{R_3 T R^2}{|Z_3|^2}\right] \quad \dots \quad (32)$$

and $$F = \frac{4R_g R_L}{G}\cdot\frac{1}{R_g}\cdot\frac{T}{T_0}\left[R_g\frac{T_0}{T} + R_1 + \frac{R_2 R^2}{4\omega_2^2 L^2 |I_p|^2} + \frac{R_3 R^2}{|Z_3|^2}\right] \quad \dots \quad (33.a)$$

From equation. (20) we have

$$G = \frac{4R_g R_L}{|Z_{11}|^2} = \frac{4R_g R_L}{(|Z_1|-R)^2}$$

Therefore,

$$F = 1 + \frac{T}{T_0}\left[\frac{R_1}{R_g} + \frac{R_2}{R_g}\left(\frac{R}{2\omega_2 L|I_p|}\right)^2 + \frac{R_3}{R_g}\left(\frac{R}{|Z_3|}\right)^2\right] \quad \dots \quad (33.b)$$

Equation. (33.b) can be rewritten as

$$\left[(F-1)\frac{T_0}{T} - \frac{R_1}{R_g}\right]\frac{R_g R_3}{R_T^2} = \frac{R^2}{R_T^2}\left[\frac{1}{Q_2 Q_3\left(\frac{\omega_2}{\omega_3}\frac{4L^2}{L_2 L_3}\right)|I_p|^2} + 1\right] \quad \dots \quad (33.c)$$

Denoting the left hand side of equation. (33.c) by η, we have

$$\eta = \frac{R^2}{R_T^2}\left[\frac{1}{Q_2 Q_3\left(\frac{\omega_2}{\omega_3}\cdot\frac{4L^2}{L_2 L_3}\right)|I_p|^2} + 1\right] \quad \dots \quad (33.d)$$

The variation of η with pump current I_p is depicted in figure 7.

Before proceeding to analyse the next case, it is worthwhile to note that if we use non-linear capacitance as shown in figure 4, instead of non-linear inductance, the method of analysis will remain unchanged. Thus, if we assume that the voltage across the coupling capacitor is given by

$$V_{(t)} = \int[S_0 i_{(t)} + S i^2_{(t)}]dt \quad \dots \quad (34)$$

where, $S = 1/C$, we can write any one of the above relations simply by replacing $2j\omega L$ by $S/j\omega$ or $1/j\omega c$ in the relations derived for inductance coupling case. For

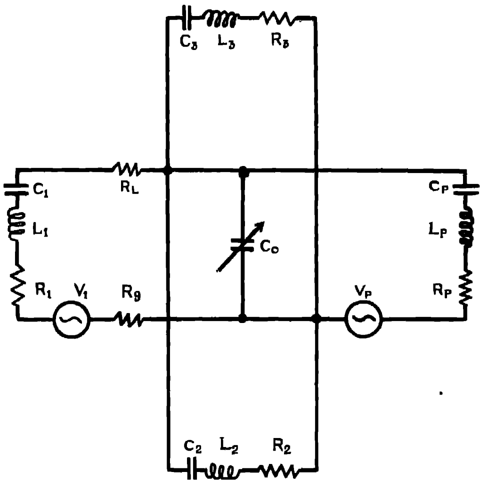

Fig. 4. Schematic representation of a single pump two idler parametric amplifier having a variable non-linear capacitance as the coupling element.

example, the expression for effective self impedance of the signal circuit will be given by

$$Z_{11} = Z_1 + \left[\frac{\dfrac{S^2 \mid I_p \mid^2}{\omega_1 \omega_2}}{Z_2 - \dfrac{S^2 \mid I_p \mid^2}{\omega_2 \omega_3 z_3^*}} \right] \quad \ldots \quad (35)$$

III. ANALYSIS OF CASE II

(1) *Phase relations* : Consider the combination of a mixer and an amplifier employing two pumps and two idlers. The frequency relations in this case are

$$\omega_1 = \omega_2 + \omega_p \qquad \text{(mixer)} \qquad (36.a)$$

$$\omega_q = \omega_2 + \omega_3 \qquad \text{(amplifier)} \qquad (36.b)$$

The pumping source at frequency ω_q supplies power to ω_2 and ω_3. The power at ω_2 mixes with that at ω_p to give power at ω_1. An analysis similar to case I will give following phase relations;

$$\theta_3 = \theta_q - \theta_2 + \pi/2 \qquad (37)$$

and

$$\theta_1-\theta_p=\theta_2+\pi/2 \quad \text{and} \quad I_q'I_3' > I_1'I_p' \qquad \ldots \quad (38.a)$$

or

$$\theta_1-\theta_p=\theta_2-\pi/2 \qquad \ldots \quad (38.b)$$

In this case too, although the amplification of the signal will be achieved if the phase relations are those as given by equation (38.a), the possibility of attenuation is not ruled out and the signal will suffer attenuation if equation (38.b) instead of (38.a) holds good.

(2) *Power relations* The power relations are given by equations. (39)

$$P_1 = 2\omega_1 L \mid I_1 I_2 I_p \mid \qquad \ldots \quad (39.a)$$

$$P_2 = -2\omega_2 L[\mid I_1 I_2 I_p \mid + \mid I_2 I_3 I_q \mid] \qquad \ldots \quad (39.b)$$

$$P_3 = -2\omega_3 L \mid I_2 I_3 I_q \mid \qquad . \quad (39.c)$$

$$P_p = -2\omega_p L \mid I_1 I_2 I_p \qquad \ldots \quad (39.d)$$

$$P_q = 2\omega_q L \mid I_2 I_3 I_q \mid \qquad \ldots \quad (39.e)$$

(3) *Negative resistance* : Proceeding exactly in the same way as in case I one can write the effective self impedance of the signal circuit as

$$Z_{11} = \frac{V_1}{I_1} = Z_1 - \frac{4\omega_1'\omega_2'L^2 \mid I_p \mid^2}{\dfrac{4\omega'_2\omega'_3 L^2 \mid I_q \mid^2}{Z_3^*} - Z_2} \qquad \ldots \quad (40)$$

The negative resistance and reactance, in general, are given by the following equations ·

$$R = \frac{R_T Q_1 Q_2 \beta_{1p}\beta_{2p}\left(Q_2 Q_3 \beta_{2q}\beta_{3q} - 4Q_3^2 \dfrac{\Delta\omega^2}{\omega_3^2} - 1\right)}{\left(Q_2 Q_3 \beta_{2q}\beta_{3q} + 4Q_2 Q_3 \dfrac{\Delta\omega^2}{\omega_2\omega_3} - 1\right)^2 + 4\Delta\omega^2 \left(\dfrac{Q_2}{\omega_2} + \dfrac{Q_3}{\omega_3}\right)^2} \qquad \ldots \quad (41.a)$$

$$X = 2R_T\Delta\omega \left[\frac{Q_1 Q_2 \beta_{1p}\beta_{2p}\left(\dfrac{Q_2}{\omega_2} + \dfrac{Q_2 Q_3^2 \beta_{2q}\beta_{3q}}{\omega_3} + \dfrac{4Q_2 Q_3^2 \Delta\omega^2}{\omega_2\omega_3^2}\right)}{\left(Q_2 Q_3 \beta_{2q}\beta_{3q} + 4Q_2 Q_3 \dfrac{\Delta\omega^2}{\omega_2\omega_3} - 1\right)^2 + 4\Delta\omega^2 \left(\dfrac{Q_2}{\omega_2} + \dfrac{Q_3}{\omega_3}\right)^2}\right] \ldots \quad (41.b)$$

At resonance

$$R_{res} = \frac{R_T Q_1 Q_2 \beta_{1p}\beta_{2p}}{Q_2 Q_3 \beta_{2q}\beta_{3q} - 1} \qquad \ldots \quad (41.c)$$

where

$$\beta_{kq} = \frac{2L}{L_k} \cdot |I_q| \frac{\omega_k'}{\omega_k} \qquad \dots \quad (42)$$

The variation of fractional negative resistance at resonance is depicted in figure 5.

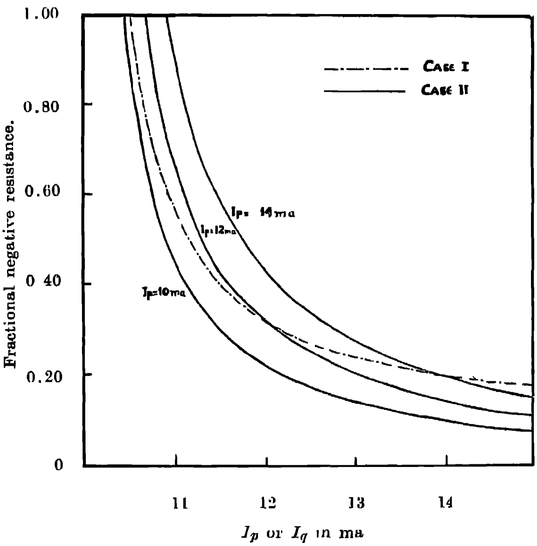

Fig. 5. Plots of variation of fractional negative resistance with pump currents, assuming that,

$$10Q_1 \frac{2L}{L_1} = Q_2 \frac{2L}{L_2} = Q_3 \frac{2L}{L_3} = 100/mA$$

It is clear from the expression of the negative resistance at resonance that the same value of gain or negative resistance can be achieved by a number of combinations of I_p and I_q. Writing

$$\gamma_1 = Q_1Q_2\beta_{1p}\beta_{2p} \quad \text{and} \quad \gamma_2 = Q_2Q_3\beta_{2q}\beta_{3q}, \text{ we have}$$

$$\frac{R_{res}}{R_{q'}} = \alpha = \frac{\gamma_1}{\gamma_2 - 1}$$

Hence $$\gamma_2 = 1 + \frac{\gamma_1}{\alpha} \quad \text{or} \quad \frac{\gamma_2}{\gamma_1} = \frac{1}{\gamma_1} + \frac{1}{\alpha} \qquad \dots \quad (43)$$

This relation is depicted in figure 6. It is easy to see from figure 6 that the large values of negative resistance are obtained by choosing small values of the ratio $\frac{\gamma_2}{\gamma_1}$. It is also obvious that $\frac{\gamma_2}{\gamma_1}\left(\text{or } \frac{I_q^2}{I_p^2}\right)$ is smaller, the larger is γ_1(or I_p^2). Therefore, the condition of large gain demands a small value of the ratio $\frac{I_q^2}{I_p^2}$ and a large value of I_p^2.

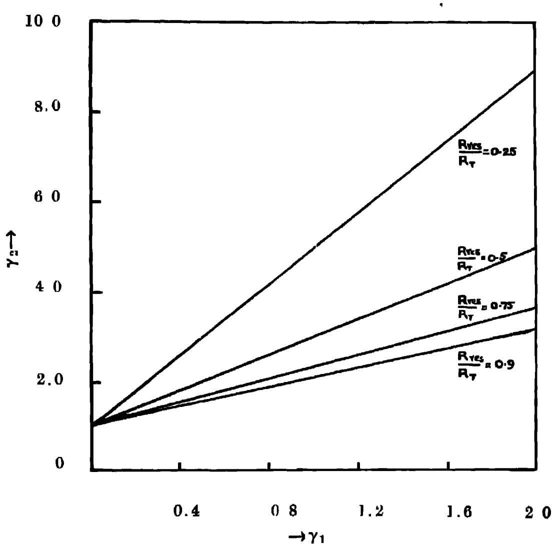

Fig. 6. Plots of variation of γ_2 with γ_1 for different values of fractional negative resistance [see equation (43)].

(4) *Gain and band-width* : The expressions for gain of a parametric amplifier employing a combination of a mixer and an amplifier is not much different from the previous case. We have, in this case

$$G = \frac{4R_gR_L}{(R_T - R)^2 + (X_1 - X)^2} \qquad \dots \quad (44)$$

and

$$\frac{G}{G_{max}} = \frac{(R_T - R_{res})^2}{(R_T - R)^2 + (X_1 - X)^2} \qquad \dots \quad (45)$$

The band-width will be equal to the difference of the roots of the equation.

$$\frac{G}{G_m} \quad \frac{1}{2} \qquad \ldots \quad (46.a)$$

that is,

$$2\left[1 - \frac{Q_1 Q_2 \beta_{1p} \beta_{2t}}{Q_2 Q_3 \beta_{2q} \beta_{3q} - 1}\right]^2 R_T^2 = (R_T - R)^2 + (X_1 - X)^2 \qquad \ldots \quad (46b)$$

(5) *Noise figure* : The ultimate noise figure is analogous to that of case I, and can be written as

$$F = 1 + \frac{T}{T_0}\left[\frac{R_1}{R_g} + \frac{R_2}{R_g}\left(\frac{R}{2\omega_2 L\,|I_p|}\right)^2 + \frac{R_3}{R_g}\left(\frac{R}{|Z_3|}\right)^2\left(\frac{|I_q|}{|I_p|}\right)^2\right] \qquad (47)$$

The equation (47) can be rewritten as

$$\left[(F-1)\frac{T_0}{T} - \frac{R_1}{R_g}\right]\frac{R_g R_3}{R_T^2} = \frac{R^2}{R_T^2}\left[\frac{1}{Q_2 Q_3\left(\frac{\omega_2}{\omega_3}\cdot\frac{4L^2}{L_2 L_3}\right)|I_p|^2} + \left(\frac{|I_q|}{|I_p|}\right)^2\right] \ldots \quad (48a)$$

or

$$\eta = \frac{R^2}{R_T^2}\left[\frac{1}{Q_2 Q_3\left(\frac{\omega_2}{\omega_3}\cdot\frac{4L^2}{L_2 L_3}\right)|I_p|^2} + \left(\frac{|I_q|}{|I_p|}\right)^2\right] \ldots \quad (48.b)$$

The variation of η with pump currents is depicted in figure 7.

The expression (48) indicates that the noise figure can be reduced by choosing a small value of the ratio $\frac{\omega_3}{\omega_2}$ and of $\frac{I_q}{I_p}$. It should be noted that the requirements of a small value of I_q/I_p and a large value of I_p demanded by equation (48) for a small noise figure are consistent.

IV. ANALYSIS OF CASES III & IV

(1) *Phase relations*: For the combination of two mixers and an amplifier the frequency relations are

$$\omega_1 = \omega_2 + \omega_p \qquad \text{(mixer)} \qquad \ldots \quad (49.a)$$

$$= \omega_3 + \omega_q \qquad \text{(mixer)} \qquad \ldots \quad (49.b)$$

$$\omega_q = \omega_2 + \omega_3 \qquad \text{(amplifier)} \qquad \ldots \quad (49.c)$$

ω_2 and ω_3 receive powers from the pumping source ω_q and after mixing with ω_p and ω_q respectively, deliver power at the frequency ω_1 with the result that the signal power is amplified.

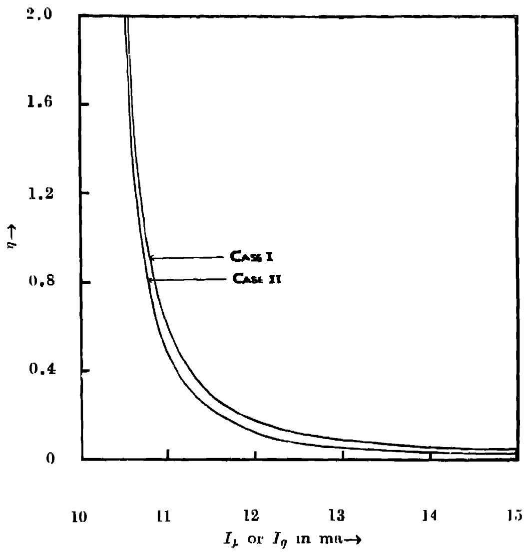

Fig. 7. Plots of $\eta = \left[(F-1)\ \frac{T_0}{T} - \frac{R_1}{R_g}\right] \frac{R_g R_3}{R_T{}^2}$ with pump currents, assuming that,

$$Q_2 Q_3 \left(\frac{\omega_2}{\omega_3} \cdot \frac{4L^2}{L_2 L_3}\right. \quad 10{,}000/(mA)^2$$

In order to achieve amplification of the signal one of the following phase relations should be satisfied.

(a)

$$\left.\begin{aligned} \theta_3 &= \theta_1 - \theta_q + \frac{\pi}{2} \\ &= \theta_q - \theta_2 + \frac{\pi}{2} \\ \theta_2 &= \theta_1 - \theta_p - \frac{\pi}{2} \end{aligned}\right\} \quad \ldots \quad (50.a)$$

and $\qquad I'_2 I'_p > I_3' I'_q > I_1' I'_p$

(b)
$$\begin{aligned}\theta_3 &= \theta_1 - \theta_q - \frac{\pi}{2}\\ &= \theta_q - \theta_2 + \frac{\pi}{2}\\ \theta_2 &= \theta_1 - \theta_p - \frac{\pi}{2}\end{aligned} \qquad (50.b)$$

and $I_3'I_q' > I_1'I_p'$ and $I_2' > I_1'$

(2) *Power relations*. The power relations are given by the following equations.

$$P_1 = 2L\omega_1[\,|I_1I_2I_p| + |I_1I_3I_q|\,] \quad \ldots \quad (51.a)$$

$$P_2 = 2L\omega_2[\,|I_1I_2I_p| + |I_2I_3I_q|\,] \quad \ldots \quad (51.b)$$

$$P_3 = 2L\omega_3[\,|I_1I_3I_q| + |I_2I_3I_q|\,] \quad \ldots \quad (51.c)$$

$$P_p = -2L\omega_p|I_1I_2I_p| \quad \ldots \quad (51.d)$$

$$P_q = 2L\omega_q[\,|I_2I_3I_q| - |I_1I_3I_q|\,] \quad \ldots \quad (51.e)$$

(3) *Negative resistance* The negative resistance at resonance is given by equation. (52)

$$R_{res} = \frac{4\omega_1\omega_2L^2|I_1|^2R_3 + 4\omega_1\omega_3L^2|I_q|^2R_2 \pm |16\omega_1\omega_2\omega_3L^3|I_p|\,|I_q|^2}{\omega_2\omega_3L^2|I_q|^2 - R_2R_3} \quad \ldots \quad (52.a)$$

$$= \frac{R_T[Q_1Q_2\beta_{1p}\beta_{2p} + Q_1Q_3\beta_{1q}\beta_{3q} \pm 2Q_1Q_2Q_3\beta_{1p}\beta_{2q}\beta_{3q}]}{[illegible]} \quad \ldots \quad (52.b)$$

The negative and the positive signs correspond to phase conditions given by equationns. (50.a) and (50.b) respectively. Therefore, the amplification will be greater when the phase conditions are those given by equation. (50.b).

CASE IV.

(1) *Phase relations* : This is a multi-idler case and employs a combination of a mixer and two amplifiers. The frequency relations are as given below:

$$\omega_1 = \omega_2 + \omega_p \qquad \text{(mixer)} \quad \ldots \quad (53.a)$$

$$\omega_p = \omega_2 + \omega_3 \qquad \text{(amplifier)} \quad \ldots \quad (53.b)$$

$$\omega_q = \omega_3 + \omega_4 \qquad \text{(amplifier)} \quad \ldots \quad (53.c)$$

The idlers ω_2 and ω_4 receive powers from pumping sources ω_p and ω_q respectively, while the idler ω_3 receives power from both the pumps. ω_2 mixes with ω_p to give power at frequency ω_1 and thus the amplification of the signal results.

The phase relations resulting in amplification are given below.

$$\left.\begin{aligned} \theta_4 &= \theta_q-\theta_3+\frac{\pi}{2} \\ \theta_3 &= \theta_p-\theta_2+\frac{\pi}{2} \\ \theta_2 &= \theta_1-\theta_p-\frac{\pi}{2} \\ \text{and} \qquad I_3' &> I_1' \end{aligned}\right\} \qquad \dots (54)$$

(2) *Power relations* : The powers at various frequencies entering the reactor are given by the following equations .

$$P_1 = \quad 2L\omega_1[\,|I_1I_2I_p|\,] \qquad \dots (55.a)$$

$$P_2 = -2L\omega_2[\,|I_1I_2I_p|+|I_2I_3I_p|\,] \qquad \dots (55.b)$$

$$P_3 = -2L\omega_3[\,|I_2I_3I_p|+|I_3I_4I_q|\,] \qquad \dots (55.c)$$

$$P_4 = -2L\omega_4[\,|I_3I_4I_q|\,] \qquad \dots (55.d)$$

$$P_p = -2L\omega_p[\,|I_1I_2I_p|-|I_2I_3I_p|\,] \qquad \dots (55.e)$$

$$P_q = \quad 2L\omega_q[\,|I_2I_3I_q|\,] \qquad \dots (55.f)$$

(3) *Negative resistance* : The self impedance Z_{11} in this case is given by

$$Z_{11} = Z_1 - \cfrac{\omega_1'\omega_2'L^2|I_p|^2}{\cfrac{\omega_2'\omega_3'L^2|I_q|^2}{Z_3^* - \cfrac{\omega_3'\omega_4'L^2|I_q|^2}{Z_4}} - Z_2} \qquad \dots (56)$$

At resonance the negative resistance is

$$R_{res} = -\cfrac{\omega_1\omega_2L^2|I_p|^2}{\cfrac{\omega_2\omega_3L^2|I_q|^2}{R_3 - \cfrac{\omega_3\omega_4L^2|I_q|^2}{R_4}} - R_2} \qquad \dots (57.a)$$

$$= R_T \cfrac{Q_1Q_2\beta_{1p}\beta_{2p}}{\cfrac{Q_2Q_3\beta_{2q}\beta_{3q}}{1-Q_3Q_4\beta_{3q}\beta_{4q}} - 1} \qquad \dots (57.b)$$

In concluding it would be good to compare the expressions of fractional negative resistance of all the parametric amplifiers so far evolved. The expression of the fractional negative resistance for single pump and single idler as well as two pumps and single idler case (Bloom and Chang, 1958) using third order non-linearity are given below.

(i) $$R_{res} = \frac{\omega_1\omega_2 L^2 |I_p|^2}{R_2} \quad \ldots \quad (58)$$

(ii) $$R = \left[\frac{3}{2} L \left| \frac{V_{p1}}{Z_{p1}} \right| \left| \frac{V_{p2}}{Z_{p2}} \right| \right]^2 \frac{R_d}{R_d^2+X_d^2}\, \omega(p_1+p_2-\omega) \quad \ldots \quad (59.a)$$

Corresponding expressions for different cases considered in this paper are to be found in equations (19.b), (41.c), (52 b) and (57.b).

It will be observed that the negative resistance in case of multi-idler circuit is less than that obtained in other cases and therefore, they offer no added advantage. The third case viz. the lower frequency pumping parametric amplifier considered as a combination of two mixers and one amplifier using two pumps and two idlers, seems to be the best one with regard to gain.

Writing expression (59.a) in terms of equivalent inductance and according to the symbols used in this text we have,

$$R_{res} = \frac{1}{4}\, \frac{L_p^2 L_q^2 \omega_1 \omega_2}{R_2} \quad \ldots \quad (59.b)$$

$$= \frac{1}{4}\, R_T Q_1 Q_2 \beta_{1p}\beta_{2q} \quad \ldots \quad (59.c)$$

Remembering the equation (27) it is easily seen that for the same values of I_p and I_q the negative resistance obtained in our combinations is greater than that obtained in the above one [vide equation (59.c)]. We, therefore, anticipate that the gain will also be larger in the present combinations.

V. AN ALTERNATIVE METHOD OF ANALYSIS

If the coupling reactor is assumed varying at the pump frequency, that is, if the inductance $L_{(t)}$ is given by equation (60), the analysis will remain unaltered except that we have to substitute L_p for $-2LI_p$ and L_q for $-2LI_q$ in each of the equations. The inductance $L_{(t)}$ is given by

$$L_{(t)} = L_p[e^{j(\omega_p t + \theta_p)} + e^{-j(\omega_p t + \theta_p)}] \quad \ldots \quad (60.a)$$

for one pump case, and

$$L_{(t)} = L_p[e^{j(\omega_p t + \theta_p)} + e^{-j(\omega_p t + \theta_p)}] + L_q[e^{j(\omega_q t + \theta_q)} + e^{-j(\omega_q t + \theta_q)}] \qquad \ldots \quad (60.b)$$

for two pump case.

ACKNOWLEDGMENTS

The authors wish to thank Prof. K. Banerjee, Head of the Department of Physics, University of Allahabad and Dr. S. N. Ghosh, Professor, Institute of Applied Physics, University of Allahabad, for their constant encouragement and keen interest. One of the authors (Dikshit) extends thanks to the Government of India for the award of Senior Research Training Scholarship.

REFERENCES

Bloom, S., and Chang, K. K. N., 1957, *R. C. A. Rev.*, **18**, 578.

Heffner, H., and Wade, G., 1958, *J. App. Phy.*, **29**, 1321.

Chang, K. K. N. and Bloom, S., 1958, *Proc. I. R. E.*, **46**, 1383

Bloom, S. and Chang, K K N., 1958, *J. App. Phy.*, **29**, 594.

Manley, J. M. and Rowe, H. E., 1956, *Proc. I. R. E.*, **44**, 904.

Hogan, C. L., Jepsen, R. L. and Vartanian, P. H., 1958, *J. App. Phy.*, **29**, 422.